AN INVESTIGATION INTO GRADE 12 TEACHERS’ UNDERSTANDING OF EUCLIDEAN GEOMETRY

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ABSTRACT:

The purpose of this study was to investigate Grade 12 mathematics teachers’ understanding of Euclidean Geometry. This study was guided by the qualitative method within an interpretive paradigm. A case study was undertaken using a written test and a task-based interview. The Bloom’s Taxonomy learning domain and the Van Hiele Theory of levels of Thought in Euclidean Geometry were used as the theoretical framework for this study. Teachers demonstrated limited understanding of Bloom’s Taxonomy category 4 through to category 5, the Van Hiele level 3 through to level 4 and non-routine problems. These findings raised questions about their ability and competency if questions and problems go a little bit beyond the textbook and non-routine examination questions.

Keywords: Mathematics, Euclidean Geometry, Bloom’s Taxonomy, Van Hiele, Teachers’ understanding.

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INTRODUCTION:

In my teaching experience of mathematics, the part that is the most difficult for learners is Euclidean Geometry. According to Mthembu (2007) it is often felt by stakeholders that learners are weak and not the teachers. Consequently this study specifically investigated teachers’ understanding of Euclidean Geometry with specific reference to circle geometry. It is premised on the assumption that if teachers do not understand Euclidean Geometry, they are not likely being able to teach it properly. Through traditional Euclidean Geometry has been omitted from the mathematics curricula of many countries; it is experiencing a revival (De Villiers, 1997 & DBE, 2011). It is believed by some that the life skills honed by such geometric reasoning remain relevant, whatever the mathematics curriculum (Van Putten et al., 2010).

As from 2008, Euclidean Geometry in its traditional forms of the theorem recognition, solving riders and proofs construction has been optional in the South African matriculation exam papers. It is now part of the optional Mathematics Paper 3, which most schools do not teach. Out of ten teachers who participated in this study, only one has been teaching the optional third mathematics paper. The implementation of Curriculum and Assessment Policy Statement (CAPS) for Grade 10-12 Mathematics (DBE, 2011) in South Africa is currently under way. CAPS was implemented in Grade 10 last year 2012 onwards as the year progress, and Euclidean Geometry will again form part of the second mathematics paper and; therefore, there will be no more optional papers. Only two papers will be written in Grade 10 as from year 2012 onwards.

One of the main reasons why Euclidean Geometry was made optional in South Africa in 2008 was the view that teachers
are not familiar with the content (Bowie, 2009). Some teachers find the Euclidean Geometry section difficult, even if they studied Euclidean Geometry in high school and at tertiary level; let alone those who did not study Euclidean Geometry in high school or at tertiary level. Therefore, the question is whether the main reason for making it voluntary has actually been resolved or not. In this regard, this study specifically focused on investigating a sample of mathematics teachers’ understanding of Euclidean Geometry to identify areas of strengths and weaknesses and hopes to make recommendations on how to develop their understanding in their areas of weakness.

Teachers’ understanding was investigated according to Bloom’s Taxonomy levels of assessment and Van Hiele levels of understanding Euclidean Geometry. Bloom’s Taxonomy has six categories of assessment levels, while Van Hiele has five levels of understanding Euclidean Geometry. This study focused on the first five categories of Bloom’s Taxonomy and the first four levels of Van Hiele theory. The research was structured around the following questions:

1. What are the general Bloom’s Taxonomy learning domains and Van Hiele levels of understanding of Euclidean Geometry of a sample of Grade 12 mathematics teachers?
2. What are the sample of Grade 12 mathematics teachers’ specific misconceptions with respect to circle geometry?

Theoretical Framework

This study is underpinned by a version of a newly revised Bloom’s Taxonomy categories and the Van Hiele levels of understanding geometry theory. These two models can be used as teaching tools as well as assessment tools. These models provided a conceptual framework that facilitated an understanding of the topic for this study.
Bloom’s Taxonomy was named after Benjamin S. Bloom. He was at the forefront of educational theory in United States in the 1950’s. In 1956, eight years after the committee of the college first began, work on the cognitive domain was completed and handbook commonly referred to as “Bloom’s Taxonomy” was published (Forehand, 2010). Bloom’s categories are:

Category 1 – Remembering; Category 2-Understanding;
Category 3 – Applying; Category 4 - Analysing;
Category 5 – Evaluating and Category 6 – Creating.

The Van Hieles were two Dutch educators who experienced challenges in their own classroom trying to categorise students’ thinking in geometry by levels. The so-called Van Hiele theory originated in the respective doctoral dissertations of Dina Van Hiele-Geldof and her husband Pierre Van Hiele at the University of Utrecht, Netherlands in 1957. The Van Hiele thinking/reasoning levels are:

Level 1 – Visualisation; Level 2 - Analysis; Level 3 - Ordering/Abstract; Level 4 – Deduction and Level 5 – Rigor.

The main reason that led to the failure of traditional geometry curriculum was attributed by the Van Hieles to the fact that the curriculum was presented at a higher reasoning level than those of the pupils (De Villiers, 2010). Monaghan (2000) assumes that most learners start a high school geometry course thinking at the first or second level. The Van Hiele theory has a distinction of five levels in the mastery of geometry, and the hypothesis that they form a learning hierarchy. The first four levels are the most pertinent ones for secondary school geometry and level five is pertinent at tertiary level geometry. The theory states that someone cannot be at a specific level without having passed through the preceding levels.

This study was based on Subject Matter Knowledge (SMK) of mathematics teachers especially those who did
Euclidean Geometry during their schooling and taught it in Grade 12. Mathematics teachers’ competence and understanding of Euclidean Geometry will determine how they will teach it. Therefore mathematics teachers should be at least a level ahead of learners for cognitive levels (Pandiscio & Knight, 2009). Bloom’s Taxonomy categories and Van Hiele theory are constructive attempts to identify persons’ stage of geometric understanding as well as the means to progress through the categories and levels. Progression through the Van Hiele levels is dependent more on the type of instruction received than on the persons’ physical or biological maturation level (Pandiscio & Knight, 2009). The categories and theory if applied appropriately throughout the schooling phase will result in geometric thinking becoming accessible to all.

Method

This study was designed to investigate Grade 12 teachers’ level of understanding of circle geometry in Euclidean Geometry. The method employed for completion of this study is the qualitative research method. This investigation therefore lends itself to qualitative method within an interpretive paradigm, since the researcher attempted to understand this in terms of teachers’ understanding of circle geometry according to new revised Bloom’s Taxonomy Categories of learning levels and the Van Hiele Theory rather than generalising the results.

The researcher collected data by giving participants a test to write. To supplement the information gathered from the test, participants were also given a task-based interview. This study collected data in two forms from the person, in the form of written and spoken words (test and task-based interview). For this reason, a qualitative research approach was used in this study. Conducting a test and analysing the results by means of a task-based interview allowed the researcher to understand teachers’
intentions and reflections better. The test was firstly administrated to five Grade 12 teachers from Zululand District previously known as Vryheid District in KwaZulu-Natal before the main participants wrote it and a task-based interview was conducted to the same population to increase validity of the questions in the test and task-based interview. The final questions were then produced after some adjustments were made, based on the pilot study.

The test was designed and developed by the researcher, with questions posed in English. Individual items in the test were designed or adapted by the researcher to comply with the requirements of this study. The test consisted of twelve questions, based on the first five categories of Bloom’s Taxonomy and the first four levels of Van Hiele Theory. It was not the aim of this study to test the validity of Bloom’s Taxonomy Categories and the Van Hiele Theory since these were reliably done (Gutierrez et al., 1991; Senk, 1989 and Mayberry, 1983), however, it was better to use both as a tool in the attempt to explore the status of geometric understanding in teachers with specific reference to circle geometry.

In this study, participants were interviewed individually in order to understand individual levels of insight based on Bloom’s Taxonomy Categories and Van Hiele theory. Teachers completed the given test on their own, which was then followed by the task-based interview. The purposes for the teachers to complete the test were twofold. Firstly, the strategies which teachers employ to solve geometric riders, with specific reference to circle geometry, can be ascertained. Secondly, using the teachers’ responses to the test, the researcher was able to identify/formulate task-based interview questions for each participant individually.
In this study, the target population is being described as all Grade 12 mathematics teachers in Umkhanyakude District previously known as Obonjeni District. Since all participants could not be reached because of time constrains and other logistics problems, a representative sample was selected. Teachers who participated in this study were from three different wards yet within the same district, therefore the findings will be assumed to be most representative to the entire district. The targeted population was ten teachers, who have ten or more years experience in teaching Grade 12 mathematics irrespective of their qualifications. The district has more than 172 mathematics teachers, since mathematics is offered in 172 high schools within Umkhanyakude District.

The permission to conduct research in the KwaZulu-Natal Department of Basic Education (KZN DBE) institutions and ethical clearance from University of KwaZulu-Natal was obtained. Consent forms were obtained from all the participants before the commencement of the study and they granted permission for the task-based interviews to be audio taped.

Results

Shulman (2004) argues that teachers must be in possession of deep SMK and understanding; therefore, teachers must be above learners’ SMK in terms of Bloom’s Taxonomy categories and the Van Hiele levels. Teachers, also, need to understand the central concepts of circle geometry and understand how best to present and communicate specific concepts, in order to overcome misconceptions. Similarly, in his discussion, Freire (1989) as cited in Howard & Aleman (2008) points out the need for teachers to be knowledgeable in their field to be able to apply a challenging curriculum. The data captured show that teachers lack this type of knowledge, especially in high order or non-routine
questions. Out of twelve questions, I will discuss only three questions, one which is non-routine problem.

QUESTION FOUR (BTC 2 AND VHL 2)
Represent the following statement with suitable rough sketches (drawings), as you might do in class with learners.
“The angle subtended by a chord at the centre of a circle is equal to twice the angle subtended by the same chord at the circle.”

This question is based on the theorem know as “angle at centre is twice angle at circumference” in Euclidean Geometry curricula at school level. Seemingly theorems are presented as a finished product. In Grade 11 where this theorem is taught, some text-books represent it in three different cases (like Classroom Mathematics for Grade 11 and Version 1 CAPS Grade 11 Mathematics), therefore the researcher expected teachers to sketch three drawings to represent the three different cases. As it is shown on the table above only one teacher (10%) of the teachers represented the given theorem in the three different cases, others used less than three sketches.

Discussion of teachers answers of Question Four
Out of 6 teachers who obtained inconclusive responses with some attempt, 5 of them sketched only one diagram. This question required plural sketches (drawings) to represent the different cases of the theorem. It therefore appeared that the majority of these teachers only expose their learners to the standard representation of this theorem (e.g. the 2nd one in Figure 1), which is likely to seriously disadvantage their learners should they need to recognise and apply the theorem in one of the other two possible cases.

A correct test answer (Teacher TMM)
Teacher TMM shows an understanding of this theorem in different representations; and the teacher represented it in three different ways. However, in his first representation, he wrote the fourth example of the theorem and often missed by learners $\frac{1}{2} \angle EOC = \angle EBC$ which is not given by Laridon, but an example of this theorem.

**Figure 1: Response by teacher TMM**

*Inconclusive test response (Teacher TLD)*

Teacher TLD as shown in Figure 2 only drew the standard representation, but tried to find other representations but failed. This question constitutes Bloom’s Taxonomy category 2 and Van Hiele level 2, where teachers start to analyse technical terminology to interrelate figures or properties of figures.

**Question Four:**
Represent the following statement with suitable rough sketch (drawing), as you might do in class with learners.

The angle subtended by a chord at the centre of a circle is equal to twice the angle subtended by the same chord at the circle.

**Figure 2: Response by teacher TLD**
**Incorrect test response (Teacher TWD)**

Teacher TWB as shown in Figure 3 obtained an incorrect response in this question. He confused the theorem with another theorem known as “perpendicular from centre to chord”, and one can speculate that it might be due to carelessness or some language difficulty since is an English second language speaker. Either way, this is problematic for a Grade 12 teacher if s/he is not to even be able to carefully read the formulation of a theorem and represent different cases.

![Figure 3: Response by teacher TWB](image)

**Response by teacher TWB**

**QUESTION FIVE (BTC 3 & 5 and VHL 2 & 3)**

5.1. If angle $A$ was a right-angle, would $ABCD$ necessarily be a square?

5.2. Justify your answer by logical reasoning or providing a suitable counter example.
This question required teachers to know properties of a square, which are:

- All angles are equal to 90°; and
- All sides are equal.

For teachers to obtain correct response with valid reasons and working shown, they needed to have some understanding of necessary and sufficient conditions for a square.

**A correct test answer, but no justification (Teacher TLD)**

According to data collected, teacher TLD gave the correct response in 5.1, but could not justify why $ABCD$ is not necessarily a square, as shown in Figure 4. This teacher applied given information to answer 5.1 but could not justify it.

**Figure 4: Response by teacher TLD**

**A correct test answer and justification (Teacher TTN)**

Teacher TTN’s response is given in Figure 5 and shows an understanding of the conditions for a square when justifying test question 5.1 response. The teacher considered what is given to build up her logical argument. The teacher further declared that, even if $\angle A = \angle C = 90^\circ$, that does not mean $\angle D$ will be $90^\circ$, but $\angle B$ and $\angle D$ can be any degree, the key is that they will be supplementary.
Teacher TDM’s response is given in Figure 6. In contrast to teacher TTN, teacher TDM thought that it would be a square, although teacher TDM also arrived at the correct conclusion that \(\angle B + \angle D = 180^\circ\), but then incorrectly deduced that this implies that \(\angle B + \angle D = 90^\circ\). It is, however, not clear why teacher TDM thought this to be the case and it might have been interesting to pursue this aspect further during an interview.

Teacher TWB also thought \(ABCD\) would be a square and tried to justify his response logically as shown in Figure 7 similar to teacher TDM, teacher TWB seemed to also assume that if two opposite angles in a cyclic quadrilateral are equal to 90° each, then, that will mean that the other two angles are 90° also, marking all four sides equal.
Figure 7: Response by teacher TWB

It is worrying that six out of ten teachers were initially unable to give a correct response, not realising that there was insufficient information to logically conclude that $ABCD$ was a square. This shows a lack of propositional reasoning located at Van Hiele level 3. They seemed to have difficulty in either logically determining that insufficient information was given or an inability in simply providing a counter-example, such as a rectangle. According to the curriculum, learners ought to learn how to refute false statements, but textbooks traditionally do not provide such experience to learners. They only have to accept given statements as true or to prove they are true. Moreover, as this example shows teachers themselves do not appear competent in refuting statements. This seems to be an area that might need attention in the mathematics education of future mathematics teachers, and for the professional development of current mathematics teachers.

Question Twelve (BTC 5 and VHL 4)

Prove that $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F = 360^\circ$

This problem was chosen as it is not a routine textbook problem, to check whether teachers would be able to solve a non-routine problem that they have not seen before by analysing and applying their knowledge of circle geometry. One would expect good geometry teachers to be able to solve problems that they have not seen before, irrespective of whether they teach
Mathematics Paper 3 or not. Teachers needed to construct a structure or a pattern from diverse elements, and put parts together to form a whole with emphasis on creating a new meaning or structure. At this level, teachers started to develop longer sequences of statements and began to understand the significance of deduction and the role of axioms. The diagram provided is a cyclic hexagon of which the two sums of alternate angles have to be proven to be equal to 360 degrees.

**An inconclusive test answer (Teacher TMM)**

Teacher TMM constructed radii, which did not assist in attaining a conclusive response to the question, except showing that the angle at a centre summed up to 360° revolution.

![Diagram of a cyclic hexagon](image)

An inconclusive test answer (Teacher TMM)

The diagram provided is a cyclic hexagon of which the two sums of alternate angles have to be proven to be equal to 360 degrees.

**Figure 8: Response by teacher TMM**

Inconclusive task-based interview answer (Teacher TMB)

The majority of teachers did not demonstrate an understanding of this question even though they were given a hint to draw a diagonal to divide the hexagon into two cyclic quadrilaterals. Most teachers constructed lines that did not assist them in answering this question. None of the nine interviewed teachers interviewed afterwards achieved a correct response for this question, as shown in Figure 9.
Figure 9: Response by teacher TMB

A correct test answer (Teacher TSD)

Only teacher TSD demonstrated an understanding in attaining a conclusive response to this question. This is the only participant who achieved a correct response with valid reasons in this question.
Figure 9: Response by teacher TSD

The researcher of this study observed that most teachers performed poorly or did not answer non-textbook and non-routine examination questions. The ability and competence of this study population raised questions since they are experienced Grade 12 mathematics teachers.

Level of geometric knowledge

For the teacher to demonstrate understanding of each category or level, the researcher decided on the criterion that one should obtain 60% or more in that specific categories or level. The researcher introduced BTC 0 and VHL 0, for teachers that could not obtain 60% and above in category 1 or level 1 used in this study.

According to Figure 14, two teachers could not even obtain the lower category (BTC 1). Two other teachers achieved only category 2 as their highest category. This is worrying since these teachers have an experience of ten years or more and teaching Grade 12. However, every teacher is expected to use
Bloom in designing assessment tasks (DoE, 2003), meaning they ought to be familiar with categories. Moreover, one would expect a Grade 12 learner at least to master Bloom’s Taxonomy category 3 questions.

Four out of ten teachers obtained category 3 as their highest category, including one teacher who did not obtain the previous categories. This may have been the result that category 3 (eight items) had more testing items than other categories. With few items findings cannot be generalised, except to be directly drawn to the sample. Two teachers obtained category 4, including one teacher who did not obtain category 2 and 3. None of the participants in this sample obtained category 5; the results may be that two out of three testing items were non-routine problems or non-examination questions. Only two teachers were outliers and eight teachers followed a hierarchy to achieve the highest Bloom’s Taxonomy category. In summary, eight teachers demonstrated an understanding of BTC 0 through BTC 3, two in BTC 4 and none in BTC 5.

The Van Hiele Theory, deals particularly with geometric thought as it progress over numerous levels of complexity underpinned by school curricula. Two out of ten teachers obtained VHL 0, meaning they could not achieve even the lower level. It is alarming for a Grade 12 mathematics teachers to be unable obtain 60% or above in visualising testing items. One out of eight teachers obtained level 1, as his highest level.

Two teachers obtained level 2 as their highest level, including one outlier. Testing items for Van Hiele level 2 were more based on properties of geometric shapes and logical reasoning of the responses, but it only involved three out of nineteen items. Four teachers achieved level 3, including an outlier that did not obtain level 1. This may be the result that level 3 has nine out of nineteen testing items and included secondary school curriculum problems.

One teacher obtained level 4 hierarchy and he is currently teaching Mathematics Paper 3. Two out of three items at this level were non-routine, typical questions asked in Senior SA
Mathematics Olympiad or AMESA Mathematics Challenges at provincial level. Despite several years of teaching experience, the majority of these teachers could only attempt standard problems within textbooks or curriculum of their grades. An important hypothesis of the Van Hiele Theory is that the levels form a hierarchy (De Villiers, 2010); one cannot be at a specific level without having passed through the preceding levels. However, two out of ten are outliers that are not in agreement with the theory.

**Geometric misconceptions**

This study is formed by 12 questions, with 19 items in total testing Bloom’s Taxonomy category 1 through category 5 and Van Hiele level 1 through level 4. Specific misconceptions with respect to circle geometry were identified in all Bloom’s Taxonomy categories and Van Hiele levels used in this study. In two questions teachers demonstrated adequate reasoning and understanding but some misconceptions were identified, that resulted in inconclusive responses. For example, seven out of ten teachers showed a specific conception in visually estimating their drawn angles in one; several teachers did not draw it reasonably close to 90 degrees.

**DISCUSSION**

Based on this case study, Grade 12 mathematics teachers in Umkhanyakude District do not have an in-depth understanding of geometric concepts. It is expected that teachers leaving matric as students having successfully completed the mathematics geometry attained Van Hiele level 3 and leaving tertiary course having attained Van Hiele level 4. The overall test results, however, show that teachers attained below 60% on Van Hiele level 2 through to level 4. More than half of these teachers were efficiently functional on level 1. According to the Van Hiele Theory, a person who is situated below level 4 can do proofs in no way other than memorisation (Mthembu, 2007). These teachers lack understanding of geometry, since most of them obviously functioned no higher than upper reaches of level 3.
The test and task-based interview in this study revealed that general mathematics teaching experience do not enhance mathematical thinking skills on its own in terms of Bloom’s Taxonomy categories and Van Hiele levels. This study found that the majority of teachers, even after teaching calculus, were still functioning well on Van Hiele level 1 compared to any other level. Despite the fact that teachers had completed several years of secondary school geometry courses, having studied under the old school curriculum (like NATED 550) in which Euclidean Geometry was compulsory and they taught it prior 2008, they still lack understanding of upper levels. However, visualisation and thinking skills cannot be ignored in this study since they are prerequisite for the upper levels.

When teachers were informed about this study, majority showed excitement; however, during task-based interview they uniformly expressed fear of Euclidean Geometry in general. The comments of prospective teachers who participated in Pournara (2004) reflected a limited view of Euclidean geometry, while they are expected to teach the subject in the not too distant future. Perhaps even more frightening to contemplate are those who did not learn Euclidean Geometry in both secondary school and at tertiary level, while they are expected to teach it in the implementation of CAPS in 2012. The task-based interview results showed that there was a general improvement in all levels and categories; however, some teachers still could not answer level 1 (visualisation) questions.

Van Hiele level 4, which is the desired level of competence for mathematics teachers in South African secondary schools, had not been uniformly achieved till the end of this study. The poor response to non-routine problems raised questions about the ability and competence of this sample of teachers if it can go a little bit beyond the textbook and non-routine examination questions. One can only wonder, somewhat fearfully, what the situation is like among the total population of Grade 10 – 12 teachers in South Africa.
CONCLUSION:

The intention of this study was to make a meaningful contribution to the body of knowledge related to teachers’ understanding of Euclidean Geometry. Some of the teachers become frustrated, demotivated and indifferent towards Euclidean Geometry because they feel incompetent in dealing with it. Researchers such as De Villiers (1997) have argued strongly and presented evidence that geometry as a research area within mathematics is alive and well. Euclidean Geometry at school is now also experiencing a renaissance in South Africa in the implementation of CAPS and other countries, at all levels of education.

For the curriculum reform initiatives to be of any significance, there need to be a radical re-look at the teachers education courses at both preservice and in-service levels. Most high school teachers do not teach Mathematics Paper 3. Higher institutions of learning offering teacher education courses need to have compulsory modules in Euclidean and non-Euclidean Geometry for both primary and secondary teachers.

The researcher believes that such a study would be able to inform effective teaching Euclidean Geometry. There is a need to understand geometry teaching practise at the chalk face/phase: how teachers teach geometry, how they use the language of geometry, and to investigate the extent to which their use of the language of geometry takes into consideration learners’ level of development in terms of the Van Hiele theory. We need to explore further to an extent to which providing pre-service and in-service teachers with opportunities to engage in activities that “require classifying answers by Van Hiele Levels” (Feza and Webb, 2005, p. 45) might contribute to effective practice in Euclidean Geometry.

REFERENCES:


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